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3D polynomial dynamical systems with elementary first integrals

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Abstract

Here we present a semi-algorithm to find elementary first integrals of 3D polynomial dynamical systems. It is a Darboux type procedure that extends the method built by Prelle and Singer for 2D systems. Although it cannot deal with the general case, the method presents a direct/simple way to find elementary first integrals.

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1. Introduction

Recently, there has been an increase in the search for algorithmic methods to deal with dynamical systems. One of the main reasons is that the recent advances in computational power allow a fast calculation of many procedures that were beyond all possibilities a few years ago.

The problem involving the integrability of dynamical systems is classical and difficult. In essence, this would imply finding a certain number of first integrals. Many of the most effective and used methods are based on ansatz on the general structure of such first integrals, and therefore, not algorithmic in nature. Darboux, in 1878 [1], gave the first steps in order to algorithmically determine first integrals, basing his method on a link between algebraic geometry and the search of first integrals. He showed how to construct the first integrals of a planar polynomial vector field having sufficient number of invariant algebraic curves. Those algebraic curves are defined by polynomials: the so-called Darboux polynomials. In general, the most complex task involved in any Darboux type method is the determination of the Darboux polynomials themselves.

In [2], Cairó and Llibre pointed out that by using the exponential factors, introduced by Cristopher [3], the Darboux methods could be generalized to deal with elementary (rather than just algebraic) first integrals.

Concomitantly to these developments, in 1983, Prelle and Singer [5] found a semi-algorithmic approach to find elementary first integrals of 2D vector fields. The attractiveness of the Prelle–Singer (PS) method lies not only in the fact that it is based on an algebraic point of view but also if the given system has a first integral in terms of elementary functions, the method guarantees that this first integral will be found (though, in principle it can admittedly take an infinite amount of time to do so).

Because of its remarkable characteristics the PS method has generated many extensions [6–18]. In particular, in [8–10, 12, 13, 15] the PS method was extended to deal with rational first-order ordinary differential equations (ODEs) presenting Liouvillian solutions. In [16–18], the PS method was extended to deal with second-order ODEs. These approaches dealt with rational second-order ODEs that present elementary[1] general solutions or elementary first integrals.

Our idea in this work was to extend the PS method for 3D polynomial dynamical systems. In this paper, we present a semi-algorithm that extends the work done in [5] to find elementary first integrals of a class of 3D polynomial systems of first-order ODEs and is a natural extension of [18][2]. In addition this semi-algorithm can also find first integrals that were missed by the method of exponential factors introduced by Cairó and Llibre [2][3].

In section 2, we present some basic concepts and a summary of the work by Prelle and Singer. In the following section, we introduce our main results and how to use them to produce a semi-algorithm to find elementary first integrals of a class of 3D polynomial systems of first-order ODEs. Finally, we present our conclusions and point out some directions to further our work.

2. Some basic concepts and the PS method

In this section, we will describe some basic concepts involving 2D polynomial systems of first-order ODEs and present some results of Prelle and Singer that were necessary to extend their method.

2.1. Integrating factors for 2D autonomous systems of first-order ODEs

Let us consider a system given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = N(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = M(x, y), \tag{1}$$

where M and N are polynomials in (x, y).

A function $I(x, y)$ is a *first integral* of system (1) if $I(x, y)$ is a constant over all solution curves of (1), i.e. $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$. Therefore, over the solutions,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \partial_x I \dot{x} + \partial_y I \dot{y} = N\partial_x I + M\partial_y I = 0. \tag{2}$$

Defining

$$D \equiv N\partial_x + M\partial_y, \tag{3}$$

as the *Darboux operator* associated with the 2D system (1), we can see that a function $I(x, y)$ is a first integral of the system *iff* $D[I] = 0$. Besides, looking at system (1) we can write it in the form

$$\mathrm{d}t = \frac{\mathrm{d}x}{N} = \frac{\mathrm{d}y}{M} \tag{4}$$

[1] For a formal definition of elementary function, see section 2.2 or [19].
[2] There we have dealt with an extension of the Prelle–Singer approach for 2ODEs.
[3] Later, in section 3.4, we will explain why.

and so, over its solutions, $M\,\mathrm{d}x = N\,\mathrm{d}y$. This means that the 1-form γ defined by $\gamma \equiv M\,\mathrm{d}x - N\,\mathrm{d}y$ is null over the solutions of the system, i.e. over the solutions,

$$\gamma \equiv M\,\mathrm{d}x - N\,\mathrm{d}y = 0. \tag{5}$$

From these results we have the following.

If I(x, y) is a first integral of the 2D system (1) then the 1-form $\mathrm{d}I$ *is proportional to the 1-form* γ *defined above, i.e.*

$$\mathrm{d}I = r\gamma, \tag{6}$$

where r is a function of (x, y).

Let us see: consider the space of 1-forms in two variables and let $\{\gamma, \nu\}$ be a basis, where γ is the 1-form defined above. So, we can write $\mathrm{d}I$ in this basis as $\mathrm{d}I = r\gamma + s\nu$ where r and s are functions of (x, y). However, from the results (2) and (5), over the solutions of system (1), we have that $\mathrm{d}I = 0$ and $\gamma = 0$. This leads to $s\nu = 0$. Since ν is independent of γ (therefore, not null over the solutions), we must have that $s = 0$. □

From (6) we have

$$I_x\,\mathrm{d}x + I_y\,\mathrm{d}y = r(M\,\mathrm{d}x - N\,\mathrm{d}y) \tag{7}$$

(where I_u means $\partial_u I$) implying that

$$I_x = rM, \qquad I_y = -rN. \tag{8}$$

Therefore, if we determine r, we can find I via quadratures:

$$\begin{aligned} I(x, y) &= \int I_x\,\mathrm{d}x + \int \left[I_y - \frac{\partial}{\partial y} \int I_x\,\mathrm{d}x \right] \mathrm{d}y \\ &= \int rM\,\mathrm{d}x - \int \left[rN + \frac{\partial}{\partial y} \int rM\,\mathrm{d}x \right] \mathrm{d}y. \end{aligned} \tag{9}$$

2.2. Three important results and the PS method

To enunciate the first result, let us make some definitions:

Definition 1. *Let K be a field of functions. The function* θ *is called an elementary generator over K if:*

(a) θ *is algebraic over K, i.e.* θ *is a solution of a polynomial equation with coefficients in K.*
(b) θ *is an exponential over K, i.e. There exists a* η *in K such that* $\theta' = \eta'\theta$*, that is another way to say that* $\theta = \exp\eta$.
(c) θ *is a logarithm over K, i.e. There exists a* η *in K such that* $\theta' = \eta'/\eta$*, that is another way to say that* $\theta = \ln\eta$.

Definition 2. *Let K be a field of functions. An extension* $E = K(\theta_1, \ldots, \theta_n)$ *is called a field of elementary functions over K if each* θ_i *is an elementary generator over K. A function is elementary over K if it belongs to a field of elementary functions over K.*

In this paper *elementary function* means a function that belongs to an elementary extension of the field $C[x, y, z]$.

Definition 3. *A function R(x, y) satisfying* $R(A\,\mathrm{d}x + B\,\mathrm{d}y) = \mathrm{d}I$ *(where A and B are functions of* (x, y)*) is called an integrating factor of the 1-form* $(A\,\mathrm{d}x + B\,\mathrm{d}y)$.

From (7), (8) and by the definition 3 above, we can see that r is an integrating factor of the 1-form $\gamma = M\,\mathrm{d}x - N\,\mathrm{d}y$. Using the compatibility condition ($I_{xy} = I_{yx}$), we get

$$Nr_x + rN_x + Mr_y + rM_y = 0. \tag{10}$$

Using the Darboux operator D defined in (3) we can write (10) in the form

$$\frac{D[r]}{r} = -(N_x + M_y). \tag{11}$$

In other words we are saying that $\frac{D[r]}{r}$ is a polynomial. This result will prove to be very important (see below).

In [5], Prelle and Singer demonstrated two other important results that, together with (11), allowed the construction of a semi-algorithm to search for elementary first integrals of 2D systems of polynomial first-order ODEs. These are as follows.

(1) Consider the system of ODEs defined by

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x_1, x_2, \ldots, x_n), \tag{12}$$

where f_i belongs to a differential field K. If there is a function $g(x_1, x_2, \ldots, x_n)$, belonging to an elementary extension of K such that g is a constant over the solutions to (12), then there is a function $I(x_1, x_2, \ldots, x_n)$ of the form

$$I = W_0 + \sum_{i>0} c_i \ln(W_i), \tag{13}$$

where W_i are algebraic functions over K, which is a constant over the solutions of (12).

(2) If the 2D system of first-order ODEs (1) has an elementary first integral then there exists an integrating factor R for the 1-form ($M\,\mathrm{d}x - N\,\mathrm{d}y$) that is an algebraic function of (x, y) such that R^κ (where κ is an integer) is a rational function of (x, y), i.e. R can be written as

$$R = \prod_i p_i^{n_i}, \tag{14}$$

where p_i are irreducible polynomials of (x, y) and n_i are non-zero rational numbers.

One can note that the first result shown above (13), pertaining the general form of the first integral, refers to systems of any order. The second result (14), on the other hand, is specific for 2D autonomous systems.

With these three results Prelle and Singer were able to construct a semi-decision procedure to find elementary first integrals of 2D autonomous systems. Let us see from (3) and (14), we have

$$\begin{aligned}\frac{D[R]}{R} &= \frac{D\left[\prod_i p_i^{n_i}\right]}{\prod_k p_k^{n_k}} = \frac{\sum_i p_i^{n_i-1} n_i D[p_i] \prod_{j\neq i} p_j^{n_j}}{\prod_k p_k^{n_k}} \\ &= \sum_i \frac{p_i^{n_i-1} n_i D[p_i]}{p_i^{n_i}} = \sum_i n_i \frac{D[p_i]}{p_i}.\end{aligned} \tag{15}$$

From (11), plus the fact that M and N are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (15) plus the fact that p_i are irreducible polynomials, we have that $p_i|D[p_i]$ (i.e. p_i is a divisor of $D[p_i]$). The irreducible polynomials p_i are called *Darboux polynomials* associated with the Darboux operator D and, since $p_i|D[p_i]$, we can write $D[p_i]/p_i = q_i$, where q_i are polynomials called *co-factors*.

We now have a criterion for choosing the possible p_i (build all the possible divisors of $D[p_i]$) and, by using (11) and (15), we have

$$\sum_i n_i \frac{D[p_i]}{p_i} = -(N_x + M_y). \tag{16}$$

If we manage to solve (16) and thereby find n_i, we know an integrating factor for the 1-form $(M\,\mathrm{d}x - N\,\mathrm{d}y)$ and the problem of finding an elementary first integral I for system (1) is reduced to a quadrature. Since we do not have an upper bound on the degree of the p_i polynomials (the building blocks of the integrating factor R), we say that the Prelle and Singer method is a semi-algorithm waiting for this upper bound to become an algorithm.

So, we can note that, basically, the method developed by Prelle and Singer is based upon three results.

(1) If R is an integrating factor of the 2D system (1), then the compatibility condition $(I_{xy} = I_{yx})$ can be written in the form $\frac{D[R]}{R} = -(N_x + M_y)$, where $D \equiv N\partial_x + M\partial_y$, i.e. $\frac{D[R]}{R}$ is a polynomial.
(2) If the 2D system (1) presents an elementary first integral, then it presents one of the type $I = W_0 + \sum_{i>0} c_i \ln(W_i)$, where W_i are algebraic functions of (x, y).
(3) If the 2D system (1) presents an elementary first integral, then it presents an integrating factor R of the form $R = \prod_i p_i^{n_i}$, i.e. R^κ (κ integer) is a rational function of (x, y).

3. A method to search for first integrals of 3D polynomial systems of first-order ODEs

In this section we will describe some basic concepts involving 3D polynomial systems of first-order ODEs and how we can produce some results that are the analog of the results presented in the end of the previous section for 2D systems. These results will allow for the production of a semi-algorithm to deal with a class of 3D polynomial systems of first-order ODEs presenting, at least, one elementary first integral.

3.1. *First integrals for 3D systems*

Consider the 3D system given by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} = f(x, y, z), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y} = g(x, y, z), \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \dot{z} = h(x, y, z), \end{aligned} \tag{17}$$

where f, g and h are polynomials in (x, y, z).

A function $I(x, y, z)$ is a *first integral* of system (17) if $I(x, y, z)$ is a constant over all solution curves of (17), i.e. $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$. Therefore, over the solutions,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \partial_x I\dot{x} + \partial_y I\dot{y} + \partial_z I\dot{z} = f\,\partial_x I + g\,\partial_y I + h\,\partial_z I = 0. \tag{18}$$

Defining now

$$D \equiv f\partial_x + g\partial_y + h\partial_z, \tag{19}$$

as the *Darboux operator* associated with (17), we can write the condition for a function $I(x, y, z)$ to be a first integral of the 3D system (17) as $D[I] = 0$. Again, like for the 2D system (1), looking at system (17) we can write it in the form

$$\mathrm{d}t = \frac{\mathrm{d}x}{f} = \frac{\mathrm{d}y}{g} = \frac{\mathrm{d}z}{h} \tag{20}$$

and so, over its solutions, we have[4] $g\,\mathrm{d}x = f\,\mathrm{d}y$ and $h\,\mathrm{d}x = f\,\mathrm{d}z$. This means that the 1-forms α and β defined by $\alpha \equiv g\,\mathrm{d}x - f\,\mathrm{d}y$ and $\beta \equiv h\,\mathrm{d}x - f\,\mathrm{d}z$ are null over the solutions of the system, i.e. over the solutions,

$$\alpha \equiv g\,\mathrm{d}x - f\,\mathrm{d}y = 0 \qquad \text{and} \qquad \beta \equiv h\,\mathrm{d}x - f\,\mathrm{d}z = 0. \tag{21}$$

From these results we have the following.

If $I(x, y, z)$ is a first integral of the 3D system (17), *then the 1-form* $\mathrm{d}I$ *is a vector in the subspace spanned by the 1-forms α and β defined above, i.e.*

$$\mathrm{d}I = r\alpha + s\beta, \tag{22}$$

where r and s are functions of (x, y, z).

Let us see: consider the space of 1-forms in three variables and let $\{\alpha, \beta, \mu\}$ be a basis, where α and β are the 1-forms defined above. We can write $\mathrm{d}I$ in this basis as $\mathrm{d}I = r\alpha + s\beta + a\mu$ where r, s and a are functions of (x, y, z). However, from the results (18) and (21), over the solutions of system (17), we have that $\mathrm{d}I = 0$, $\alpha = 0$ and $\beta = 0$. This leads to $a\mu = 0$. Since μ is not in the subspace spanned by α and β (therefore not null over the solutions), we must have that $a = 0$. □

From (22) we have

$$I_x\,\mathrm{d}x + I_y\,\mathrm{d}y + I_z\,\mathrm{d}z = r(g\,\mathrm{d}x - f\,\mathrm{d}y) + s(h\,\mathrm{d}x - f\,\mathrm{d}z) \tag{23}$$

implying that

$$I_x = -rg - sh, \qquad I_y = rf, \qquad I_z = sf. \tag{24}$$

Therefore, if we determine r and s, we can find I via quadratures:

$$\begin{aligned} I(x, y, z) &= \int I_x \mathrm{d}x + \int \left(I_y - \frac{\partial}{\partial y} \int I_x\,\mathrm{d}x \right) \mathrm{d}y \\ &\quad + \int \left\{ I_z - \frac{\partial}{\partial z} \left[\int I_x\,\mathrm{d}x + \int \left(I_y - \frac{\partial}{\partial y} \int I_x\,\mathrm{d}x \right) \mathrm{d}y \right] \right\} \mathrm{d}z \\ &= \int (-rg - sh)\,\mathrm{d}x + \int \left(rf - \frac{\partial}{\partial y} \int (-rg - sh)\,\mathrm{d}x \right) \mathrm{d}y \\ &\quad + \int \left\{ sf - \frac{\partial}{\partial z} \left[\int (-rg - sh)\,\mathrm{d}x \right.\right. \\ &\quad \left.\left. + \int \left(rf - \frac{\partial}{\partial y} \int (-rg - sh)\,\mathrm{d}x \right) \mathrm{d}y \right] \right\} \mathrm{d}z. \end{aligned} \tag{25}$$

3.2. A proposal to construct a semi-algorithm for a class of 3D polynomial systems of first-order ODEs

In this section we will construct an analog for each of the three results summarized at the end of section 2.2.

[4] We could equally well write $(g\,\mathrm{d}x = f\,\mathrm{d}y, g\,\mathrm{d}z = h\,\mathrm{d}y)$ or $(h\,\mathrm{d}x = f\,\mathrm{d}z, h\,\mathrm{d}y = g\,\mathrm{d}z)$.

3.2.1. Writing the compatibility conditions in an appropriate form. In the case of 3D polynomial systems of first-order ODEs, we have to determine two unknown functions r and s (see equations (24), (25)) if we want to obtain a first integral I by quadratures. In the case where the ratio of these two unknown functions is a rational function of (x, y, z) (this is a restriction to the general case; see the observation below in section 3.2.2), we could write the compatibility conditions in a form similar to the one presented in equation (11). This result will be presented in the following theorem.

Theorem 1. *Consider a 3D polynomial system of first-order ODEs (17), that presents an elementary first integral I. If s/r (r and s defined by equation (22)) is a rational function of (x, y, z) (i.e. $s/r = P/Q$ where P and Q are polynomials that do not have any common factors), then $f\,\frac{D[r/Q]}{r/Q}$ is a polynomial.*

Proof of theorem 1. Using the compatibility conditions ($I_{xy} = I_{yx}, I_{xz} = I_{zx}, I_{yz} = I_{zy}$) and (24) we get

$$-rg_y - r_y g - sh_y - s_y h = rf_x + r_x f, \tag{26}$$

$$-rg_z - r_z g - sh_z - s_z h = sf_x + s_x f, \tag{27}$$

$$rf_z + r_z f = sf_y + s_y f. \tag{28}$$

Performing the operations (equation (26) minus $\frac{h}{rf}$ times equation (28)) and (equation (27) plus $\frac{g}{rf}$ times equation (28)) we can write the following two equations:

$$\frac{D[r]}{r} + \left(f_x + g_y + h\frac{f_z}{f}\right) + \frac{s}{r}\left(h_y - h\frac{f_y}{f}\right) = 0, \tag{29}$$

$$\frac{D[s]}{r} + \frac{s}{r}\left(f_x + g\frac{f_y}{f} + h_z\right) + \left(g_z - g\frac{f_z}{f}\right) = 0, \tag{30}$$

where D stands for the Darboux operator associated with the 3D system (17), i.e. $D \equiv f\partial_x + g\partial_y + h\partial_z$. Since, by hypothesis, $\frac{s}{r} = \frac{P}{Q}$, where P and Q are polynomials in (x, y, z) with no common factors, we can substitute s in equations (29) and (30) by $r\frac{P}{Q}$. This will lead us to

$$fQ\frac{D[r]}{r} = -Q(ff_x + fg_y + hf_z) - P(fh_y - hf_y), \tag{31}$$

$$fP\frac{D[r/Q]}{r/Q} = -fD[P] - P(ff_x + gf_y + fh_z) - Q(fg_z - gf_z). \tag{32}$$

Adding the term $-fQ\frac{D[Q]}{Q}$ in both sides of equation (31) we have

$$fQ\frac{D[r/Q]}{r/Q} = -fD[Q] - Q(ff_x + fg_y + hf_z) - P(fh_y - hf_y). \tag{33}$$

Since, by definition, P, Q, f, g and h are polynomials and D is a differential operator with polynomial coefficients, the right-hand sides of both (32), (33) are polynomials. Therefore, in principle, $f\frac{D[r/Q]}{r/Q}$ is a rational function. So, let us represent it as $\frac{A}{B}$, where A and B are polynomials that do not have any common factors. By doing that, one can write (32), (33) schematically as

$$P\frac{A}{B} = \text{Polynomial}_1, \tag{34}$$

$$Q\frac{A}{B} = \text{Polynomial}_2. \tag{35}$$

In order to satisfy (34), (35) simultaneously, since A and B do not have common factors, it would be necessary that $B|Q$ and $B|P$. But, again by hypothesis, P and Q do not have common factors. So, one can conclude that B is a constant. Therefore, $f\frac{D[r/Q]}{r/Q} = \frac{A}{B} =$ polynomial.

□

3.2.2. Concerning the form of the first integral. As we have mentioned, the second result shown at the end of section 2.2 is valid for any number of dimensions. So, we can enunciate the following result (a corollary of a result by Prelle and Singer [5]).

If a 3D polynomial system of first-order ODEs (17) presents an elementary first integral then it possesses one of the form

$$I = W_0 + \sum_{i>0} c_i \ln(W_i), \tag{36}$$

where W_i are algebraic functions of (x, y, z).

Obs.: from (36) we can see that the partial derivatives of the first integral I (I_x, I_y, I_z) are (in general) algebraic functions of (x, y, z). So, from equations (24), r and s (and, consequently, s/r) will be algebraic functions of (x, y, z) in the general case.

3.2.3. The form of the integrating factor. To enunciate this result let us first make a definition.

Definition 4. *A function $R(x, y, z)$ satisfying*

$$R(A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z) = \mathrm{d}I = I_x\,\mathrm{d}x + I_y\,\mathrm{d}y + I_z\,\mathrm{d}z \tag{37}$$

where A, B and C are functions of (x, y, z) is called an integrating factor of the 1-form $A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z$.

Theorem 2. *Consider a 3D system of autonomous polynomial first-order ODEs (17), that presents an elementary first integral. Then, from the result by Prelle and Singer it presents one of the form $I = W_0 + \sum_{i>0} c_i \ln(W_i)$, where W_i are algebraic functions of (x, y, z). If r/s (r and s are defined above) is a rational function of (x, y, z) (i.e. $s/r = P/Q$ where P and Q are polynomials that do not have any common factors), then r/Q can be written as*

$$\frac{r}{Q} = \prod_i p_i^{n_i} \tag{38}$$

where p_i are irreducible polynomials in (x, y, z) and n_i are non-zero rational numbers.

To prove theorem 2 we will need the following lemmas.

Lemma 1. *Consider a function F of (x_1, x_2, x_3). If the differential of F can be written as*

$$\mathrm{d}F = A\left(X_1\,\mathrm{d}x_1 + X_2\,\mathrm{d}x_2 + X_3\,\mathrm{d}x_3\right), \tag{39}$$

where X_1, X_2 and X_3 are polynomial functions of (x_1, x_2, x_3) and A is an algebraic function of (x_1, x_2, x_3), then the 2D system of first-order ODEs defined as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = X_2(x_1, x_2, x_3), \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -X_1(x_1, x_2, x_3), \tag{40}$$

where x_3 is regarded as independent of t, has $F(x_1, x_2, x_3)$ as a first integral.

Proof of lemma 1. Consider the 2D system of first ODEs defined by (40). If G is a function of (x_1, x_2) such that

$$(X_2\partial_{x_1} - X_1\partial_{x_2})[G(x_1, x_2)] = 0, \tag{41}$$

then $G(x_1, x_2)$ is a first integral of (40). Applying $(X_2\partial_{x_1} - X_1\partial_{x_2})$ to $F(x_1, x_2, x_3)$, we get $X_2F_{x_1} - X_1F_{x_2}$. But, by hypothesis, $F_{x_1} = AX_1$ and $F_{x_2} = AX_2$ lead to

$$\left(X_2\partial_{x_1} - X_1\partial_{x_2}\right)[F(x_1, x_2, x_3)] = X_2AX_1 - X_1AX_2 = 0.$$

This implies that $F(x_1, x_2, x_3)$ is a first integral of (40). □

Lemma 2. *Let*

$$\frac{dx_1}{dt} = F_1(x_1, x_2), \qquad \frac{dx_2}{dt} = F_2(x_1, x_2), \tag{42}$$

be a 2D system of first ODEs, where F_1 and F_2 are polynomials in (x_1, x_2). If A_1 and A_2 $(A_1 \neq A_2)$ are integrating factors for the 1-form $F_2\,dx_1 - F_1\,dx_2$, then one of the following statements is true.

(i) $\dfrac{A_1}{A_2} = \mathcal{I}$, *where $\mathcal{I}$ is a first integral of (42).*

(ii) $\dfrac{A_1}{A_2} = c$, *where c is a constant.*

Proof of lemma 2. Let A_1 and A_2 $(A_1 \neq A_2)$ be integrating factors for the 1-form $F_2\,dx_1 - F_1\,dx_2$. Then, defining $D_{sys} \equiv F_1\partial_{x_1} + F_2\partial_{x_2}$ (see (11)), we have that

$$\frac{D_{sys}[A_1]}{A_1} = -(\partial_{x_1}F_1 + \partial_{x_2}F_2) = \frac{D_{sys}[A_2]}{A_2} \quad \Rightarrow \quad \frac{D_{sys}[A_1]}{A_1} = \frac{D_{sys}[A_2]}{A_2}.$$

Then

$$A_2D_{sys}[A_1] - A_1D_{sys}[A_2] = 0 \quad \Rightarrow \quad A_2^2D_{sys}\left[\frac{A_1}{A_2}\right] = 0 \quad \Rightarrow \quad D_{sys}\left[\frac{A_1}{A_2}\right] = 0.$$

Since $D_{sys}[A_1/A_2] = 0$, $[A_1/A_2]$ is a constant or a first integral of (42). □

Proof of theorem 2. From the result by Prelle and Singer (see section 3.2.2), if the 3D system (17) presents an elementary first integral we can assume (without loss of generality) that I is of the form $I = W_0 + \sum_{i>0} c_i \ln(W_i)$, where W_i are algebraic functions of (x, y, z). Therefore, I_x, I_y and I_z are algebraic functions of (x, y, z). In view of the fact that (see section 3.1)

$$\begin{aligned} dI &= I_x\,dx + I_y\,dy + I_z\,dz \\ &= r(g\,dx - f\,dy) + s(h\,dx - f\,dz), \end{aligned} \tag{43}$$

and f, g and h are polynomials in (x, y, z), we can infer that r and s are algebraic functions of (x, y, z). By the hypothesis of theorem 2, $\frac{s}{r}$ is a rational function of (x, y, z), so we can substitute $s = r\frac{P}{Q}$ in (43) to obtain

$$dI = \frac{r}{Q}[(-Qg - Ph)\,dx + (Qf)\,dy + (Pf)\,dz]. \tag{44}$$

Since r is algebraic and Q is a polynomial, $\frac{r}{Q}$ is an algebraic function of (x, y, z). We still have to prove that $\frac{r}{Q}$ is of form (38). To do this let us use lemma 1: comparing dI (see (44) above) and dF (see (39) on lemma 1) we have, by using lemma 1, that $I(x, y, z)$ is a first integral of the 2D system of first-order ODEs defined by

$$\frac{dx}{dt} = Qf, \qquad \frac{dy}{dt} = Qg + Ph. \tag{45}$$

We can make use of this formal result: in section 2.2 we have shown two results by Prelle and Singer. The second result, applied to the 2D system (45) states that the 1-form

$$(-Qg - Ph)\,\mathrm{d}x + (Qf)\,\mathrm{d}y \tag{46}$$

has an integrating factor R_1 of the form

$$R_1 = \prod_i {v_i}^{m_i}, \tag{47}$$

where v_i are irreducible polynomials in (x, y) and m_i are non-zero rational numbers.

However, examining $\mathrm{d}I$ (see (44)), we have that $\frac{r}{Q}$ is also an integrating factor for the 1-form (46), since

$$I_x = -\frac{r}{Q}(Qg + Ph); \qquad I_y = \frac{r}{Q}(Qf). \tag{48}$$

Since both $\frac{r}{Q}$ and R_1 are integrating factors of $(-Qg - Ph)\,\mathrm{d}x + (Qf)\,\mathrm{d}y$ we have, using lemma 2, two possibilities.

(i) $R_1 = \mathcal{I}\frac{r}{Q}$ (where $\mathcal{I}$ is a first integral of (45)).
(ii) $R_1 = c\frac{r}{Q}$ (where c is a constant).

First possibility: let us consider that $R_1 = \mathcal{I}\frac{r}{Q}$. Since $\mathcal{I}$ and I are first integrals of system (45) then $\mathcal{I} = \mathcal{F}(I)$ and, therefore, $\mathcal{I}$ is also a first integral of the 3D system (17). So, we can write

$$\mathcal{I} = \mathcal{F}(I) = \frac{R_1}{\frac{r}{Q}}. \tag{49}$$

From (49) we can see that the first integral $\mathcal{I}$ is an algebraic function of (x, y, z). This means that there is a rational first integral (see [20]) and, in this case, s and r will be rational functions of (x, y, z). This implies that $\frac{r}{Q}$ is rational and the theorem is proved.

Second possibility: let us consider that $R_1 = c\frac{r}{Q}$. In this case we can conclude that $\frac{r}{Q}$ is of the form $\prod_i {v_i}^{m_i}$, where the coefficients of the polynomials v_i could be, in principle, algebraic functions of z. However, using lemma 1 again, we have that $I(x, y, z)$ is also a first integral of the 2D system of first-order ODEs defined by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Pf, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = Qf. \tag{50}$$

Following the same reasoning above we have that there exists an algebraic integrating factor R_2 for the 1-form $(Qf)\,\mathrm{d}y + (Pf)\,\mathrm{d}z$ of the form

$$R_2 = \prod_i {u_i}^{k_i}, \tag{51}$$

where u_i are irreducible polynomials in (y, z) and k_i are non-zero rational numbers. Besides that, from equation (44) we have that

$$I_y = \frac{r}{Q}(Qf); \qquad I_z = \frac{r}{Q}(Pf), \tag{52}$$

and, so, $\frac{r}{Q}$ is an integrating factor of the 1-form $(Qf)\,\mathrm{d}y + (Pf)\,\mathrm{d}z$. Since both $\frac{r}{Q}$ and R_2 are integrating factors of $(Qf)\,\mathrm{d}y + (Pf)\,\mathrm{d}z$, we have (again by using lemma 2) two possibilities.

(i) $R_2 = \mathcal{J}\frac{r}{Q}$ (where $\mathcal{J}$ is a first integral of (50)).
(ii) $R_2 = k\frac{r}{Q}$ (where k is a constant).

The first possibility will lead to the case where we have a rational first integral and the theorem is proved. So let us consider that $R_2 = k\frac{r}{Q}$. In this case we can conclude that $\frac{r}{Q}$ is of the form $\prod_i {u_i}^{k_i}$, where the coefficients of the polynomials u_i could be, in principle, algebraic functions of x. However, we also have that $\frac{r}{Q}$ is of the form $\prod_i {v_i}^{m_i}$, where v_i are polynomials in (x, y). Therefore, $\frac{r}{Q}$ is of the form $\prod_i p_i^{n_i}$, where p_i are irreducible polynomials in (x, y, z) and n_i are non-zero rational numbers. □

From theorem 2 we can infer the following corollary.

Corollary 1. *The polynomials p_i (see theorem 2 above) are Darboux polynomials of the operator $D \equiv f\partial_x + g\partial_y + h\partial_z$ (i.e. $\frac{D[p_i]}{p_i}$ is a polynomial) or they are factors of the polynomial f.*

Proof of corollary 1. From theorem 2 we have that $\frac{r}{Q} = \prod_i p_i^{n_i}$, where p_i are irreducible polynomials in (x, y, z) and n_i are non-zero rational numbers. Since, by theorem 1, $f\,\frac{D[r/Q]}{r/Q}$ is polynomial, we have

$$f\frac{D[r/Q]}{r/Q} = f\frac{D\left[\prod_i {p_i}^{n_i}\right]}{\prod_i {p_i}^{n_i}} = f\sum n_i\frac{D[p_i]}{p_i} = \text{polynomial}, \tag{53}$$

since p_i's are irreducible, or $p_i|D[p_i]$ or $p_i|f$. □

3.3. A possible algorithm

In this section, we will make use of the mathematics constructed above in order to produce a semi-algorithm to deal with a class[5] of 3D polynomial systems of first-order ODEs.

3.3.1. The idea behind the procedure. From corollary 1 we can write

$$T \equiv r/Q = \prod_i {p_i}^{n_i} \quad \Rightarrow \quad f\frac{D[T]}{T} = f\sum n_i\frac{D[p_i]}{p_i} = f\sum n_i q_i, \tag{54}$$

where p_i are irreducible Darboux polynomials (in (x, y, z)) of the D operator and q_i are the corresponding co-factors. Then, from equations (32), (33), we can write

$$f P\frac{D[T]}{T} = f P\sum n_i q_i = -f D[P] - P(f f_x + g f_y + f h_z) - Q(f g_z - g f_z) \tag{55}$$

and

$$f Q\frac{D[T]}{T} = f Q\sum n_i q_i = -f D[Q] - Q(f f_x + f g_y + h f_z) - P(f h_y - h f_y). \tag{56}$$

Equations (55), (56) will be the basis of our procedure. Let us begin discussing our procedure by going through the main steps before formalizing the algorithm.

As it will become clear the first step of our procedure is the most costly one, i.e. time consuming. In the same way as in the Prelle–Singer procedure for 2D systems of polynomial first-order ODEs [5], determining the Darboux polynomials and corresponding co-factors for the D operator is a task that grows as the degree of such polynomials grow. Our present procedure start by performing this search (determine p_i and corresponding q_i). Assuming that this search succeeded and we found p_i and q_i for some degree deg, checking (55), (56), we see that it remains to be determined the following set of variables: $\{n_i, P, Q\}$, where n_i are constants and P and Q are polynomials. How to determine these?

[5] s/r is a rational function of (x, y, z).

Again, by inspecting (55), (56), we note that if we construct two generic polynomials P and Q (in (x, y, z))) of generic degrees $\deg_P$ and $\deg_Q$ and solve (55), (56) for the coefficients of such polynomials and for n_i, we would have found n_i, P and Q and, consequently, $T = \prod_i {p_i}^{n_i}$. Since we have $r/Q = T$, we would have found r.

Once r and s $\left(s = r\frac{P}{Q}\right)$ have been determined, we can use (25) to obtain the first integral $I(x, y, z)$ by quadratures.

3.3.2. The algorithm itself. The overview above indicates the main trust of our procedure, below we will present a step-by-step *modus operandi* of how to implement the above scenario.

- Steps of the algorithm
  1. Set $\deg = 0$.
  2. Set $\deg_Q = 0$ and $\deg_P = 0$.
  3. Increase deg: $\deg = \deg + 1$.
  4. Construct generic polynomials p and q (in (x, y, z)) of degrees deg and $MAX(\deg f, \deg g, \deg z) - 1$.
  5. Construct the operator D and determine the Darboux polynomials and the associated co-factors, up to degree deg, for the operator D.
  6. Increase $\deg_Q$ and $\deg_P$: $\deg_Q = \deg_Q + 1$ and $\deg_P = \deg_P + 1$.
  7. Construct generic polynomials Q and P (in (x, y, z)) of degrees $\deg_Q$ and $\deg_P$.
  8. Try to solve equations (55), (56). for the coefficients defining Q and P and for n_i.
  9. If we are successful, go to step 10. In the opposite case, if $\deg_Q < \deg$, we return to step 6. If $\deg_Q = \deg$, return to step 3.
  10. Check if s and r satisfy equations (26), (27) and (28). In the affirmative case go to step 11. In the negative case, return to step 6.
  11. Calculate the associated first integral using (25).

3.3.3. The inner works of the method. In this section we will present a detailed exemplification of the algorithm steps. To do so, we will apply our method to the Lorenz system, since that will permit us to see the procedure working over a very well-known system.

The Lorenz system[6]

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - xz - y, \qquad \dot{z} = xy - \beta z \tag{57}$$

is a famous dynamical model [21], where x, y and z are real variables; and s, r, and b are real parameters. This system has been thoroughly investigated as a dynamical system and, from the point of view of integrability, it was also intensively studied using different integrability theories. In [22], Llibre and Zhang provide a complete classification of the Darboux invariants of the irreducible Darboux polynomials, of the rational first integrals and of the algebraic integrability for the classical Lorenz system (see references therein).

Below, we will present in some detail the steps of our algorithm applied to the Lorenz system (we will omit the solutions with $\sigma = 0$ because in this case the Lorenz system becomes 2D).

1. $\deg = 0$.
2. $\deg_Q = 0$ and $\deg_P = 0$.
3. $\deg = 1$.
4. $p = a_1 + a_2x + a_3y + a_4z$ and $q = b_1 + b_2x + b_3y + b_4z$.

[6] $\dot{u} \equiv \frac{du}{dt}$.

5.

$$D = \sigma\,(y - x)\,\partial_x + (\rho x - xz - y)\,\partial_y + (xy - \beta z)\,\partial_z \qquad (58)$$

For deg = 1 there is only one case presenting Darboux polynomials:
$\beta = 1, \rho = 0, \sigma = \sigma$.
The Darboux polynomials and the associated co-factors are
$p_1 := y + \mathrm{i}z, q_1 := -1 + \mathrm{i}x$,
$p_2 := y - \mathrm{i}z, q_2 := -1 - \mathrm{i}x$.

6. $\deg_Q = 1$ and $\deg_P = 1$.
7. $P = p_1 + p_2 x + p_3 y + p_4 z$ and $Q = q_1 + q_2 x + q_3 y + q_4 z$.
8. FAIL.
9. deg = 2.
10. $p = a_1 + a_2 x + \cdots + a_{10} z^2$ and $q = b_1 + b_2 x + b_3 y + b_4 z$.
11. For deg = 2 there are two independent cases presenting Darboux polynomials:
    - $\beta = 2\sigma, \rho = \rho, \sigma = \sigma$.
      The Darboux polynomials and the associated co-factors are
      $p_1 := x^2 - 2\sigma z, q_1 := -2\sigma$,
      $\deg_Q = 2$ and $\deg_P = 2$.
      $P = p_1 + p_2 x + \cdots + p_{10} z^2$ and $Q = q_1 + q_2 x + \cdots + q_{10} z^2$.
      FAIL
    - $\beta = 1, \rho = \rho, \sigma = 1$.
      The Darboux polynomial and the associated co-factor are
      $p_1 := -rx^2 + y^2 + z^2, q_1 := -2$,
      $\deg_Q = 2$ and $\deg_P = 2$.
      $P = p_1 + p_2 x + \cdots + p_{10} z^2$ and $Q = q_1 + q_2 x + \cdots + q_{10} z^2$.
      FAIL
12. deg = 3.
13. $p = a_1 + a_2 x + \cdots + a_{20} z^3$ and $q = b_1 + b_2 x + b_3 y + b_4 z$.
14. For deg = 3 there are no independent solutions presenting Darboux polynomials.
15. $\deg_Q = 3$ and $\deg_P = 3$.
16. $P = p_1 + p_2 x + \cdots + p_{20} z^3$ and $Q = q_1 + q_2 x + \cdots + q_{20} z^3$.
17. FAIL
18. deg = 4.
19. $p = a_1 + a_2 x + \cdots + a_{35} z^4$ and $q = b_1 + b_2 x + b_3 y + b_4 z$.
20. For deg = 4 there are three independent solutions presenting Darboux polynomials:
    - $\beta = 4, \rho = \rho, \sigma = 1$.
      The Darboux polynomial and the associated co-factor are
      $p_1 := rx^2 + y^2 - 1/4x^4 + x^2 z - 2xy + 4z - 4zr, q_1 := -4$.
      $\deg_Q = 4$ and $\deg_P = 4$.
      $P = p_1 + p_2 x + \cdots + p_{35} z^4$ and $Q = q_1 + q_2 x + \cdots + q_{35} z^4$.
      FAIL
    - $\beta = 6\sigma - 2, \rho = 2\sigma - 1, \sigma = \sigma$.
      The Darboux polynomial and the associated co-factor are
      $p_1 := -4\frac{x^2}{s^2} + 16\frac{x^2}{s} - 16x^2 - 4\frac{x^2 z}{s} + \frac{x^4}{s^2} - 4y^2 - 8\frac{xy}{s} + 16xy$,
      $q_1 := -4\sigma$,
      $\deg_Q = 4$ and $\deg_P = 4$.
      $P = p_1 + p_2 x + \cdots + p_{35} z^4$ and $Q = q_1 + q_2 x + \cdots + q_{35} z^4$.
      FAIL

- $\beta = 0, \rho = \rho, \sigma = 1/3$.
  The Darboux polynomial and the associated co-factor are
  $p_1 := -3rx^2 + y^2 - 9/4x^4 + 3x^2z + 2xy, q_1 := -4/3,$.
  $\deg_Q = 4$ and $\deg_P = 4$.
  $P = p_1 + p_2x + \cdots + p_{35}z^4$ and $Q = q_1 + q_2x + \cdots + q_{35}z^4$.
  FAIL

21. Etc.

And so on...since, according to Llibre and Zhang [22] there are no more independent Darboux polynomials other than the ones that we have already found, if we continue indefinitely we will not succeed in finding an elementary first integral.

So, our only hope to find a first integral is that we proceed (analogously as we did in [18] for ODEs when analyzing their integrability) and include the parameters in the variables to be dealt with. In so doing, we will be dealing with sub-cases of the ones already considered above and (hopefully) we will find a working case (if they exist we will do so).

We can see, for example, that for deg $= 1$ there exists one degree of freedom in the choice of σ. We can add the parameter σ to the set of variables we want to determine. For $\deg_Q = 3$, $\deg_P = 3$ we find a solution for $\sigma = 1/2$. This leads to the following steps.

- **Step 8:** $n_1 = n_2 = 3/2, n_f = -1, p_8 = p_{30} = 1, q_9 = -1$ and $q_{29} = 1$, lead to $P = x^2z + y^2$, $Q = x^2y - yz$ and

$$R = 2\frac{1}{(y^2 + z^2)^{3/2}(x - y)}, \tag{59}$$

$$r = QR = 2\frac{(x^2 - z)y}{(y^2 + z^2)^{3/2}(x - y)}, \tag{60}$$

$$s = PR = 2\frac{x^2z + y^2}{(y^2 + z^2)^{3/2}(x - y)}. \tag{61}$$

- **Step 9:** success.
- **Step 10:** success.
- **Step 11:**

$$I = \sqrt{\frac{(x^2 - z)^2}{y^2 + z^2}}. \tag{62}$$

3.4. The contribution of our method to the scenario

In this section we will briefly discuss the contribution of our method to the Darboux type approaches. In this way, we will summarize the Darboux type procedures and comment the specificities of our algorithm with the help of an example.

As we have already mentioned, in 1878, Darboux[1] showed how to construct first integrals of 2D polynomial systems having sufficient number of invariant algebraic curves. When dealing with 3D systems, we can generalize the concept of algebraic curve. More precisely, consider the polynomial 3D system given by

$$\dot{x} = f(x, y, z), \qquad \dot{y} = g(x, y, z), \qquad \dot{z} = h(x, y, z). \tag{63}$$

where f, g and h are polynomials in (x, y, z). If p and q are two polynomials of (x, y, z) such that

$$D[p] = qp, \tag{64}$$

where $D \equiv f\partial_x + g\partial_y + h\partial_z$, the surface defined by $p(x, y, z) = 0$ is called an *invariant algebraic surface* of the system. Besides that, the polynomial q is called the *co-factor* of the invariant algebraic surface $p = 0$. On the points of an invariant algebraic surface $p = 0$, we have that $D[p] = 0$. Hence, the vector field D is contained into the tangent plane to the surface $p = 0$. So, the surface $p = 0$ is formed by trajectories of the vector field D. This justifies the name 'invariant algebraic surface' given to the algebraic surface $p = 0$ satisfying (64) for some polynomial q because it is invariant under the flow defined by D.

Basically, the method consists of performing a search of all polynomials p_i satisfying $D[p_i] = q_i p_i$. Then, one can try to find numbers n_i such that the function A defined by

$$A \equiv \sum_i p_i^{n_i} \tag{65}$$

is a first integral of the system, i.e. $D[A] = 0$.

By introducing the concept of the *exponential factor*, see Christopher [3], this makes it possible to generalize the Darboux method:

Let a, b be relatively prime polynomials in (x, y, z). Then the function $\exp(a/b)$ is called an *exponential factor* of the polynomial vector field D if the equality

$$D\left[\exp\left(\frac{a}{b}\right)\right] = K \exp\left(\frac{a}{b}\right) \tag{66}$$

is satisfied for some polynomial K of degree at most $m - 1$. As before we say that K is the co-factor of the exponential factor $\exp(a/b)$ (see [3, 4]), where the exponential factors are introduced as a limit of suitable invariant algebraic surfaces. From the point of view of the integrability of polynomial vector fields the importance of the exponential factors is twofold. On one hand, they verify (66), and on the other hand, their co-factors are polynomials of degree at most $m - 1$. These two facts allow them to play the same role as the invariant algebraic surfaces in the integrability of a three-dimensional polynomial vector field D. Christopher has also shown that if $\exp(a/b)$ is an exponential factor of the polynomial vector field D, then the polynomial b is of the form

$$b = \prod_i p_i^{n_i}, \tag{67}$$

where p_i are Darboux polynomials of the operator D and n_i are integers. That allowed Cairó and Llibre [2] to develop a method by using the exponential factors, introduced by Christopher [3]; the Darboux method could be generalized to deal with elementary (rather than just algebraic) first integrals. In this method (CLC method) one has to look for first integrals of the form

$$FI = \prod_i p_i^{\lambda_i} \prod_j \left[\exp\left(\frac{a_j}{b_j}\right)\right]^{\mu_j}, \tag{68}$$

where λ_j and μ_j are real numbers. Observing the form of the first integral I that we expect to obtain by using the CLC method, one can see that applying ln to the first integral turns it to the form

$$I = W_0 + \sum_{i>0} c_i \ln(W_i), \tag{69}$$

where W_i are rational functions of (x, y, z).

From (69) it is easy to see that the CLC method is a kind of generalization of the PS method (see section 2.2) for three dimensions. However, it misses the cases where W_i are algebraic functions of (x, y, z).

For 3D polynomial systems we could find a semi-algorithm that[7] can search for first integrals where W_i are algebraic functions of (x, y, z). Let us show an example to make things clearer. Consider the system

$$\begin{aligned}\dot{x} &= -2 + 3x^2 + 3yx - 2xz + y^2 + z^2,\\ \dot{y} &= 1 - 2x^2 - yx + 2xz - z^2,\\ \dot{z} &= x^2 - xz + zy + z^2.\end{aligned} \tag{70}$$

For this system the D operator is given by

$$\begin{aligned}D = {}&(-2 + 3x^2 + 3yx - 2xz + y^2 + z^2)\partial_x\\ &+ (1 - 2x^2 - yx + 2xz - z^2)\partial_y + (x^2 - xz + zy + z^2)\partial_z.\end{aligned} \tag{71}$$

For deg = 1, there are two Darboux polynomials. The Darboux polynomials and the associated co-factors are

$$p_1 = x + y + 1, q_1 = x + y - 1,$$
$$p_2 = x + y - 1, q_2 = x + y + 1.$$

Proceeding in according to the algorithm steps we get

$n_1 = n_2 = -3/2, n_f = -1, P = x^2 + 2xy + y^2 - 1, Q = 2x^2 + 3xy + y^2 - 2 - zx - zy,$ leading to

$$R = \frac{1}{(-2 + 3x^2 + 3xy - 2zx + y^2 + z^2)((y + x + 1)(y + x - 1))^{3/2}}, \tag{72}$$

$$r = QR = \frac{2x^2 + 3xy + y^2 - 2 - zx - zy}{(-2 + 3x^2 + 3xy - 2zx + y^2 + z^2)(x^2 + 2xy + y^2 - 1)^{3/2}}, \tag{73}$$

$$s = PR = \frac{1}{\sqrt{x^2 + 2xy + y^2 - 1}(-2 + 3x^2 + 3xy - 2zx + y^2 + z^2)}. \tag{74}$$

$$I = \frac{z + y}{\sqrt{(x + y)^2 - 1}} + \ln(x + y + \sqrt{(x + y)^2 - 1}). \tag{75}$$

Since the first integral I is non-algebraic and the system has three dimensions, it is out of the scope of the Darboux and PS methods. Besides that, applying exp to it, we obtain

$$FI = (x + y + \sqrt{(x + y)^2 - 1}) \exp\left(\frac{z + y}{\sqrt{(x + y)^2 - 1}}\right). \tag{76}$$

We can see that, since $x + y + \sqrt{(x + y)^2 - 1}$ and $\frac{z+y}{\sqrt{(x+y)^2-1}}$ are non-rational, we could not construct the first integral FI using the CLC method.

4. Conclusion

Here, we have presented a (semi)-algorithm that is a Darboux type procedure. In many ways, it is an extension (working with 3D systems) of the seminal work by Prelle and Singer [5] (dealing with 2D systems). It shares the same features as the Prelle–Singer approach: for

[7] In the case where s/r is a rational function (see section 3.2).

instance, it assures that, if the elementary first integral exists, it will eventually get it (of course, in practice it may take (depending on the order of the problem) a great deal of time to do so.

Our method is not general, it deals with the case where the corresponding first integral is elementary (please note that it is analogous to the Prelle and Singer situation where this restriction is also applied). But, when it is applicable, our algorithm proved to be very effective (in the sense that it covers some ground where other Darboux type approaches failed) and it is also very practical (in the sense that it is quick and not computationally costly).

A possible extension of the present work is to generalize these results to the case of Liouvillian first integral. This is presently being pursued.

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